

An introduction to E.S.S. and Evolutionary Games

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Workshop in the memory of Thomas Vincent

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between road traffic, message routing, Evolution theory, market analysis, breeding sheeps...

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Game of one individual against a population of identical individuals playing the same game for themselves.

Framework and notation

A large population of agents. Each has a choice of **strategies** $x \in X$.

$\Delta(X)$ is the set of positive measures of mass 1 (probabilities) over X .

$\forall A \subset X$, $n(A)$ the number of agents using a strategy $x \in A$, $n(X) = N$.
 $q(A) = n(A)/N$ the share, or *proportion*, of the population using $x \in A$.

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Three cases

- X finite, $X = \{x_1, x_2, \dots, x_n\}$, denote $q(x_i) = q_i$, $q \in \Sigma_n \subset \mathbb{R}^n$,
- $X \subset \mathbb{R}^n$, (hardly considered here), q is a measure over a continuum,
- X of infinite dimension (control). Two examples will be provided.

Strategies

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a *polymorphic* population where each agent uses always the same strategy, but collective mixed behaviour results from shares of the population choosing each strategy. (As explained above)

Fitness and generating function

Hypothesis The *fitness* that gets an agent using a strategy x is a function $G(x, q)$ of x and the *distribution of strategies* q across the population.

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Linear case

An important sub-case is when $q \mapsto G(x, q)$ is linear (a math. expectation)

$$G(x, q) = \int_X H(x, y) q(dy), \quad F(r, q) = \iint_{X \times X} H(x, y) q(dy) r(dx).$$

This is **not necessary** for many results.

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Nonlinear case

In the general case, $q \mapsto G(x, q)$ is nonlinear, we let

$$D_2 G(x, q) = H(x, y, q) , \quad D_2 G(x, q) \cdot r = \int_X H(x, y, q) r(dy) .$$

Matrix case

The *finite linear* case is called **matrix case**

$$X = \{x_1, x_2, \dots, x_n\}, \quad p, q \in \Delta(X) = \Sigma_n \subset \mathbb{R}^n,$$

$$H(x_i, x_j) = a_{ij}, \quad A = (a_{ij}).$$

$$G(x_i, p) = (Ap)_i, \quad F(q, p) = \langle q, Ap \rangle.$$

Tom Vincent's generating function

The simplicity of the matrix case with the richness of the continuous

Assume $X = [a, b] \subset \mathbb{R}$, or $X = \mathbb{R}$.

let $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, the strategies actually used

$q = (q_1, \dots, q_n) \in \mathbb{R}^n$ the proportions of the population using them

$$q(x) = \sum_i q_i \delta(x - u_i)$$

My $G(x, q(\cdot))$ is his $G(x, u, q)$.

He defines the **landscape** $x \mapsto G(x, u, q)$.

Evolutionary stability

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$$\forall q \in \Delta(X) - \{p\}, \exists \varepsilon_0 > 0 : \forall \varepsilon \in (0, \varepsilon_0), \quad F(q, q_\varepsilon) < F(p, q_\varepsilon).$$

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The supremum of such ε_0 's is called the *invasion barrier*.

Wardrop equilibrium

In the E.S.S. condition above, let $\varepsilon \rightarrow 0$. It comes

$$F(p, p) = \max_{q \in \Delta(X)} F(q, p)$$

(p, p) is a Nash point of the game $J_1(p, q) = F(p, q)$, $J_2(p, q) = F(q, p)$.

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Quote *“The journey times on all routes actually used are equal, and less than those that would be experienced by a single vehicle on any unused route [...] The first criterion is quite a likely one in practice [...] an equilibrium situation in which no driver can reduce his journey time by choosing a new route”*

John Glen Wardrop, 1952

Linear case : second order condition

The Wardrop condition is only necessary. Let the set of *best responses* to p be $B(p) = \{r \mid F(r, p) = \max_q F(q, p) = F(p, p)\}$

Proposition In the linear case, a Wardrop equilibrium p is an E.S.S. iff

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Let $X_1 = \{x \mid G(x, p) = F(p, p)\}$ and $X_2 = \text{support}(p) \subset X_1$. (W)

Let H_1 be the restriction of H to $X_1 \times X_1$ and H_2 to $X_2 \times X_2$.

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Let H_1 be the restriction of H to $X_1 \times X_1$ and H_2 to $X_2 \times X_2$.

Theorem In the linear case, a Wardrop equilibrium p is an E.S.S.

if the restriction of the quadratic form $\langle r, H_1 r \rangle$ to $r \in \mathbb{1}^\perp \subset \Delta(X_1)$ is negative definite,

only if the restriction of the quadratic form $\langle r, H_2 r \rangle$ to $r \in \mathbb{1}^\perp \subset \Delta(X_2)$ is negative definite.

Matrix case : a simple test

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 7 & 3 & 9 \\ 4 & 6 & 2 \end{pmatrix}$$

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$$\sigma(A) = \begin{pmatrix} 1 & 4 & 5 & 5 \\ 7 & -7 & 3 & 9 \\ 4 & 6 & -10 & 2 \\ 4 & 6 & -10 & 2 \end{pmatrix}$$

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A passes the test:

$$\sigma(A) + \sigma(A)^t = \begin{pmatrix} -14 & 11 \\ 11 & -20 \end{pmatrix} < 0.$$

Nonlinear case : second order condition

$H(x, y, q) = D_2G(x, q)$. Let H_1 be its restriction to $X_1 \times X_1 \times \mathcal{M}(X_1)$, and similarly H_2 its restriction to $X_2 \times X_2 \times \mathcal{M}(X_2)$.

Definition A Wardrop equilibrium is *regular* if $q \mapsto H_1(q)$ is continuous at p and the restriction of $\langle r, H_1(p)r \rangle$ to $r \in \mathbb{1}^\perp \subset \mathcal{M}(X_1)$ is negative definite.

Theorem A Wardrop equilibrium is an ESS

if it is regular, and

only if the restriction of the quadratic form $\langle r, H_2r \rangle$ to $r \in \mathbb{1}^\perp \subset \mathcal{M}(X_2)$ is nonpositive definite.

Local Superiority

Definition A strategy distribution p is called *locally superior* or equivalently p is an *Evolutionarily Robust Strategy* (E.R.S.) (or a *Neighborhood Invading Strategy* N.I.S.) if there exists a neighborhood \mathcal{N} of p such that

$$\forall q \in \mathcal{N} - \{p\}, \quad F(p, q) > F(q, q)$$

Easy result: E.R.S. \Rightarrow E.S.S. (Place q_ε in above definition and use the linearity of F w.r.t. its first argument.)

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More difficult:

Theorem In the **finite**, linear **or regular** case, E.S.S. \Rightarrow E.R.S.

Clutch size in parasitoids

Female parasitoids lay their eggs in *hosts*. 2 females parasitize each host.

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$$H = \begin{pmatrix} 1 & \pi \\ 2\pi & 0 \end{pmatrix}.$$

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$$H = \begin{pmatrix} 1 & \pi \\ 2\pi & 0 \end{pmatrix}. \quad \sigma(H) = 1 - 3\pi < 0.$$

If $\pi = 1/2$, $p_1 = 1$ and $G_i(p) = 1$.

If $\pi = 2/3$, $p_1 = 2/3$ and $G_i(p) = 8/9 < 1$.

This is an instance of Braess'paradox, well known in the transportation literature.

Replicator dynamics

Let $n(x)$ be the number (or density) of individuals using strategy x .

Let $q(x) = n(x) / \int_X dn(y)$ the strategy distribution.

Assume $G(x, q)$ is the *reproductive efficiency*. Then

Discrete generations Generation duration h

$$q(x, t + h) = q(x, t) \frac{1 + hG(x, q)}{1 + hF(q, q)}.$$

Continuous time The limit as $h \rightarrow 0$:

$$\dot{q}(x, q) = q[G(x, q) - F(q, q)].$$

Stability of the replicator equation

Theorem

- Any limit point of the replicator dynamics is a Wardrop equilibrium,
- in finite dimension, E.S.S. are Lyapunov asymptotically stable. Its attraction basin contains a neighborhood of the relative interior of the lowest dimensional face of $\Delta(X)$ it lies on.

The stability proof uses the relative entropy of q to p as Lyapunov function. Its derivative is negative if p is an E.R.S. But we have no stability result of E.R.S. in the infinite case, because that function is not weakly continuous.

A population game

Lynxes and wolves

$L \backslash W$	$cow.$	$agr.$
$cow.$	λ	0
$agr.$	$1 - \mu$	$1 - \nu$

$$\lambda + \mu > 1 > \nu$$

$$\sigma^1 = \lambda + \mu - \nu, \quad p^2 = (1 - \nu) / (\lambda + \mu - \nu),$$

$$\sigma^2 = -\lambda - \theta, \quad p^1 = \theta / (\lambda + \theta).$$

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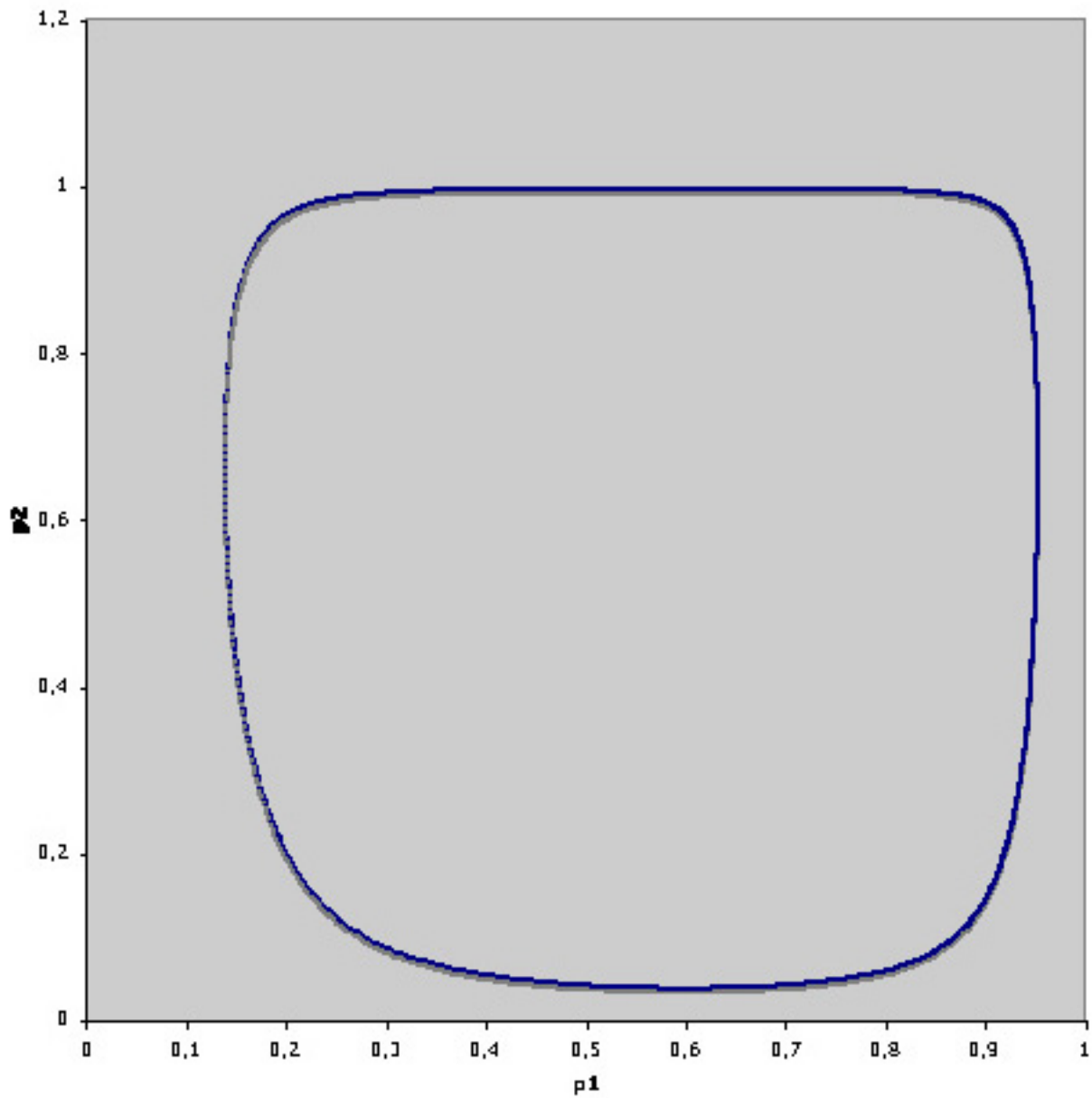
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Draw case $\lambda = \nu = 0,5, \quad \mu = 0,75, \quad \theta = 1,5.$

Wolves and Lynxes



Population games and replicator dynamics

Replicator equation of the 2×2 case: mixed Nash equilibrium (p^1, p^2) ,

$$\dot{q}^k = \sigma^k q^k (1 - q^k) (q^\ell - p^\ell), \quad \ell = 3 - k$$

Theorem If there exists a mixed Nash equilibrium,

- if $\sigma^1 \sigma^2 < 0$, the trajectories are all periodical,
- if $\sigma^1 \sigma^2 > 0$, (p^1, p^2) is a saddle. There are two (diagonally opposite) pure Nash equilibria which are asymptotically stable.

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See L. Samuelson : “Evolutionary Games and Equilibrium Selection”

Joint interest

Assume Wolves and Lynxes share a common foe : Man. Then each one benefits from the presence of the other one in repelling the foe. This creates a joint interest (similar to inclusive fitness in E.S.S.)

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Replaces the game matrices G^k by $G_{\alpha}^k = (1 - \alpha)G^k + \alpha(G^{\ell})^t$

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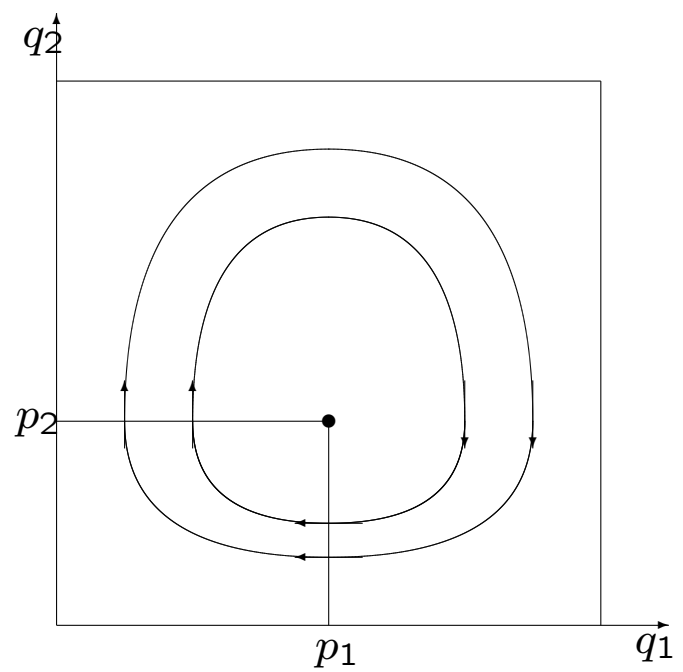
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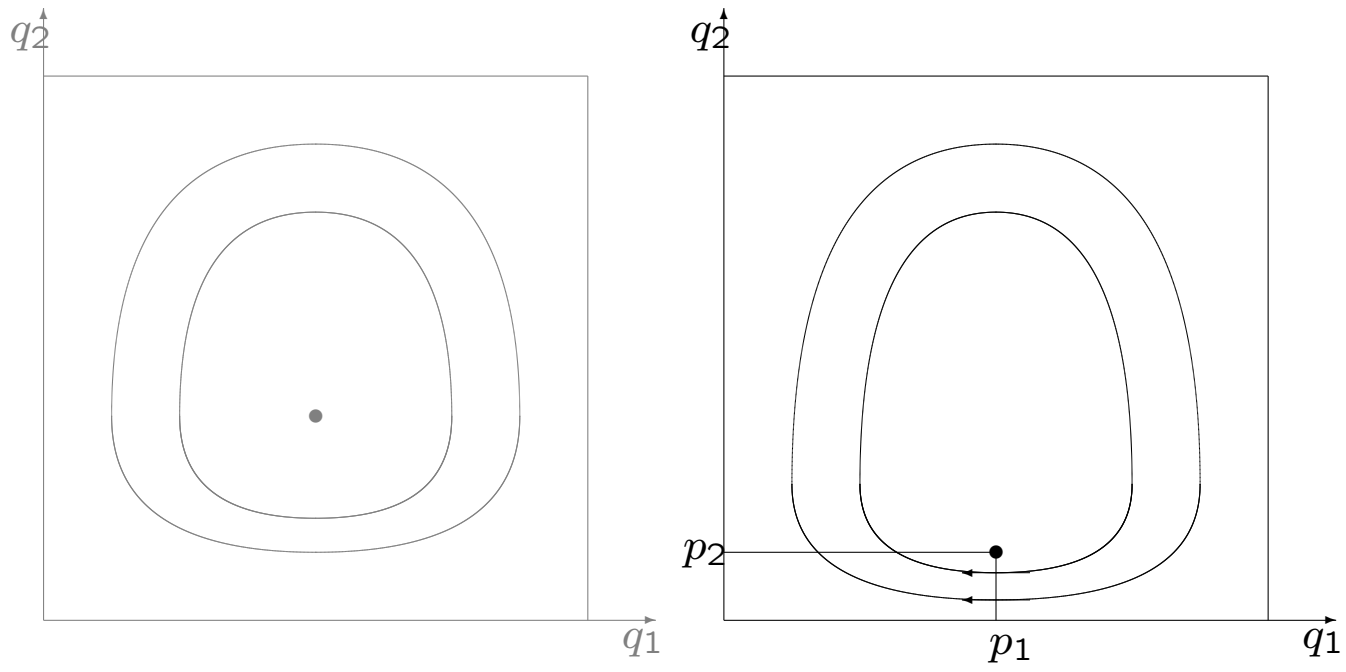
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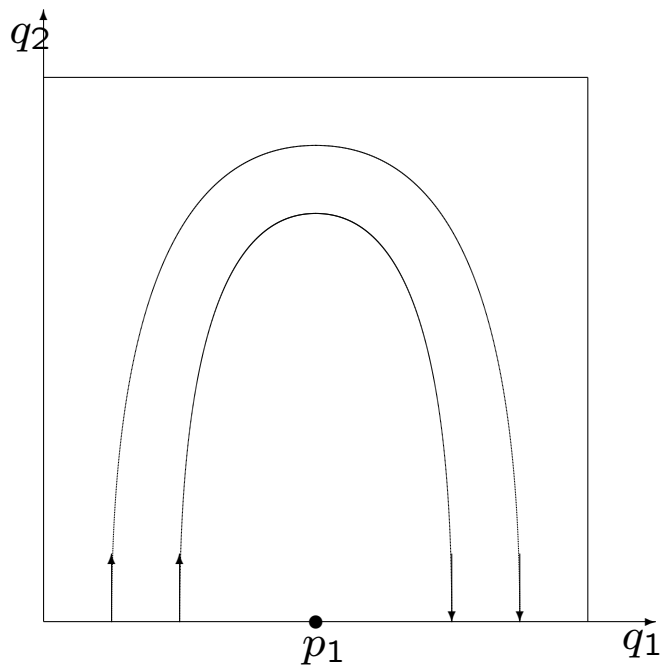
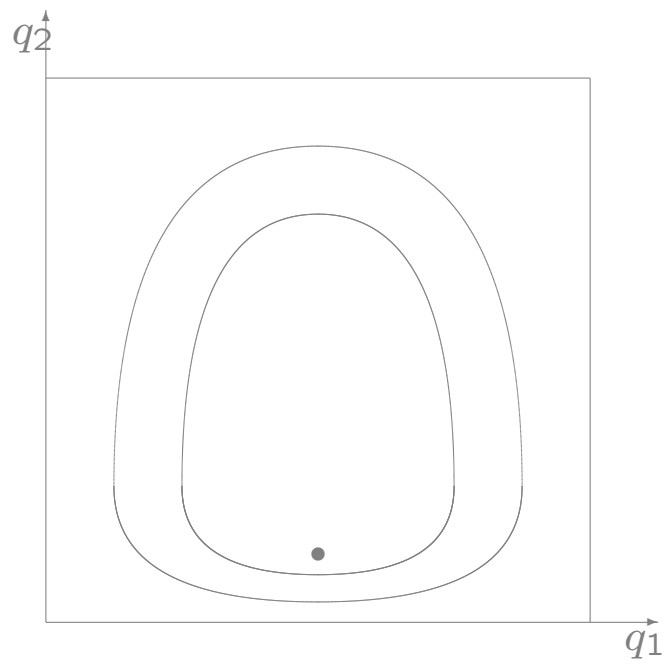
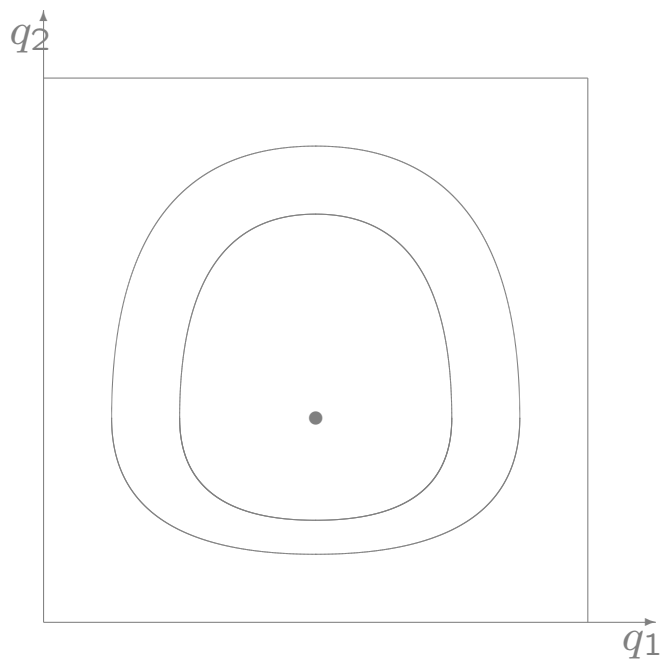
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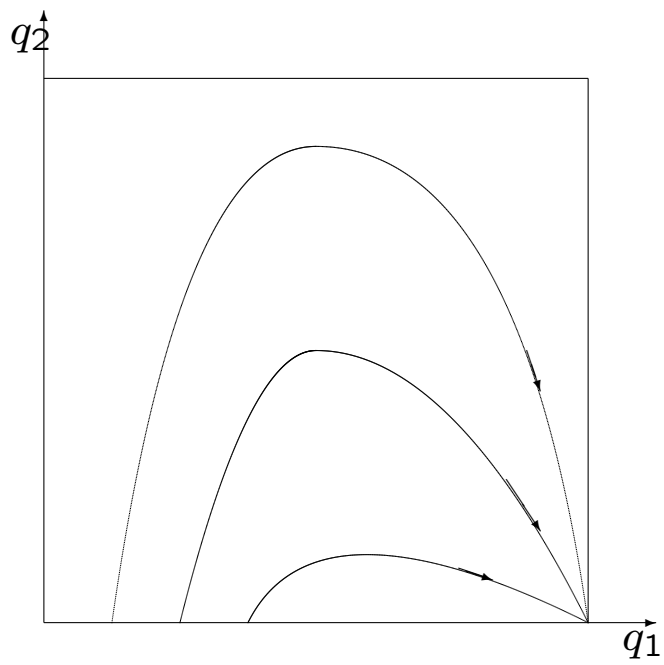
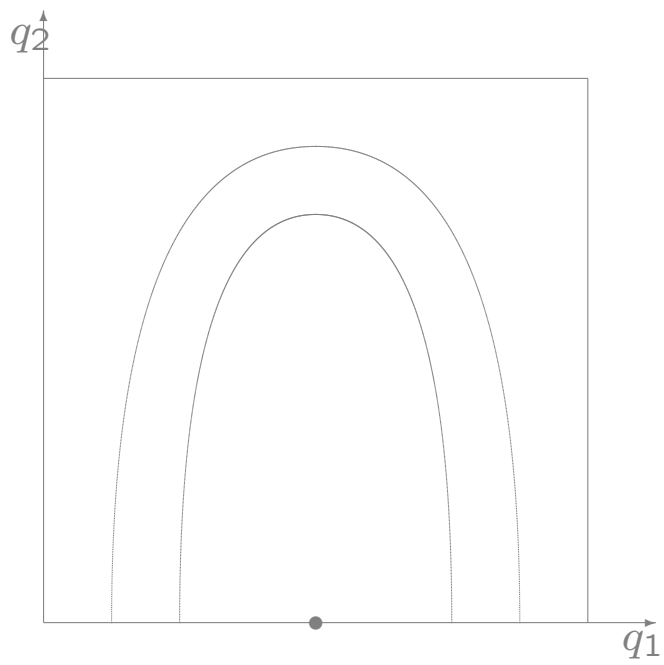
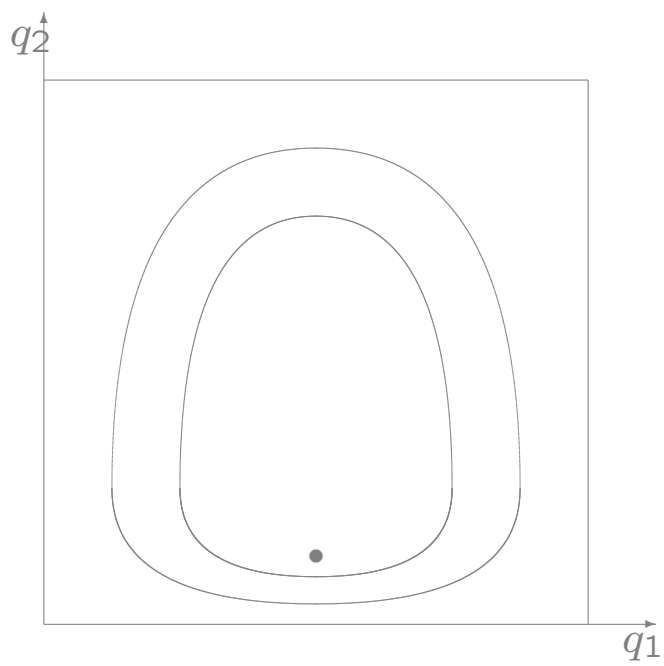
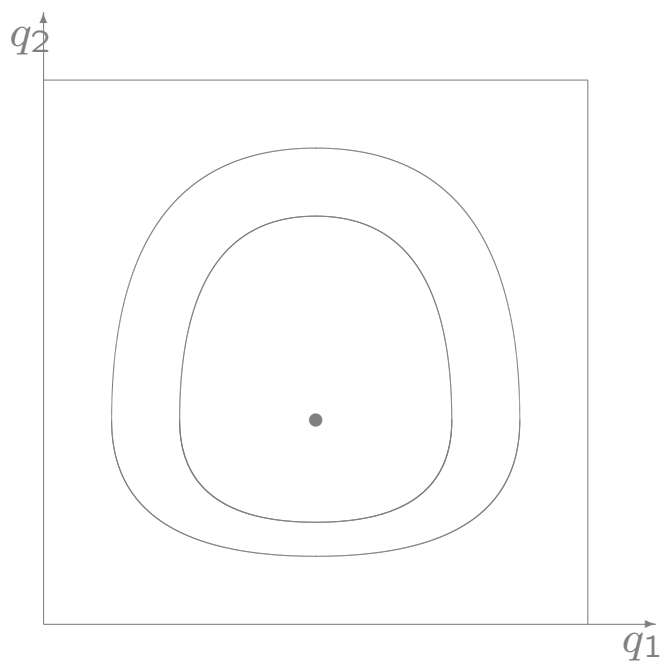
Replaces the game matrices G^k by $G_{\alpha}^k = (1 - \alpha)G^k + \alpha(G^{\ell})^t$.

A bifurcation from periodic behaviour to a stable pure strategy occurs as a p^k crosses one or zero.









Dynamics in the generating function

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x is a **control function**, X is of infinite dimension.

Bang-bang control

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Increment of fitness gained by using $x = 1$: $g(y)$. Hence

$$G(x(\cdot), q(\cdot)) = \int_0^T x(t)g(y(t)) dt.$$

Notation and hypotheses

Assume

$$Dg(y)f(y, 0) > 0, \quad Dg(y)f(y, 1) < 0.$$

Let

$$D_1f(y, q) = A(y, q), \quad D_2f(y, q) = b(y, q),$$

Hypothesis $Dg(y)b(y, q) < 0. \Rightarrow$

The equation $Dg(y)f(y, q) = 0$ generates an implicit function $q = \phi_0(y)$.

Wardrop equilibrium

Wardrop equilibrium $p(\cdot)$ generating a trajectory $z(\cdot)$ is given by $p(t) = \phi(z(t))$ where

$$\phi(y) = \begin{cases} 0 & \text{if } g(y) < 0, \\ \phi_0(y) & \text{if } g(y) = 0, \\ 1 & \text{if } g(y) > 0. \end{cases}$$

The trajectory $z(\cdot)$ reaches $\{y \mid g(y) = 0\}$ at t_0 and stays on it.

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E.S.S. ?

Necessary condition

Let $\Phi(t, s)$ be the transition matrix of $A(z(\cdot), p(\cdot))$, and

$$h(t, s) := Dg(z(t))\Phi(t, s)b(z(s), p(s))$$

Theorem A necessary condition for $p(\cdot)$ to be an E.S.S. is that

$$\forall (s, t) \in \mathcal{T} = \{s \leq t \in [t_0, T]\}, \quad h(t, t) - 2h(t, s) + h(s, s) \leq 0.$$

Proof A first order approximation of the (negative) *score function* $S(\epsilon, q) = F(q - p, q_\epsilon)$ is

$$S(\epsilon, q) = \epsilon \iint_{\mathcal{T}} (q(t) - p(t))h(t, s)(q(s) - p(s))dt ds$$

The tragedy of the Commons

The shepherds of a village share a common pasture. They may feed their flocks on the pasture ($x = 1$) or refrain ($x = 0$). The grass obeys a logistic law

$$\dot{y} = \alpha \left(1 - \frac{y}{K} \right) y + bq + c, \quad b < 0.$$

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$A(z, p) = \alpha(1 - 2\gamma/K) := a$. The test is

$$2b(e^{a(t-s)} - 1) \leq 0.$$

It succeeds if $a \leq 0$, i.e. $\gamma \in [K/2, K]$.

A routing problem (Joint work with E. Altman and A. Silva)

An ad-hoc telecommunication network densely covers an open region Ω of the plane \mathbb{R}^2 . $\Gamma = \partial\Omega = \mathcal{Q} \cup \mathcal{R}$. Messages flow in Ω through \mathcal{Q} at a given rate $\sigma(y) \geq 0$ and a density $\rho(y) \geq 0$ of messages is generated at each $y \in \Omega$. All messages have to leave through \mathcal{R} .

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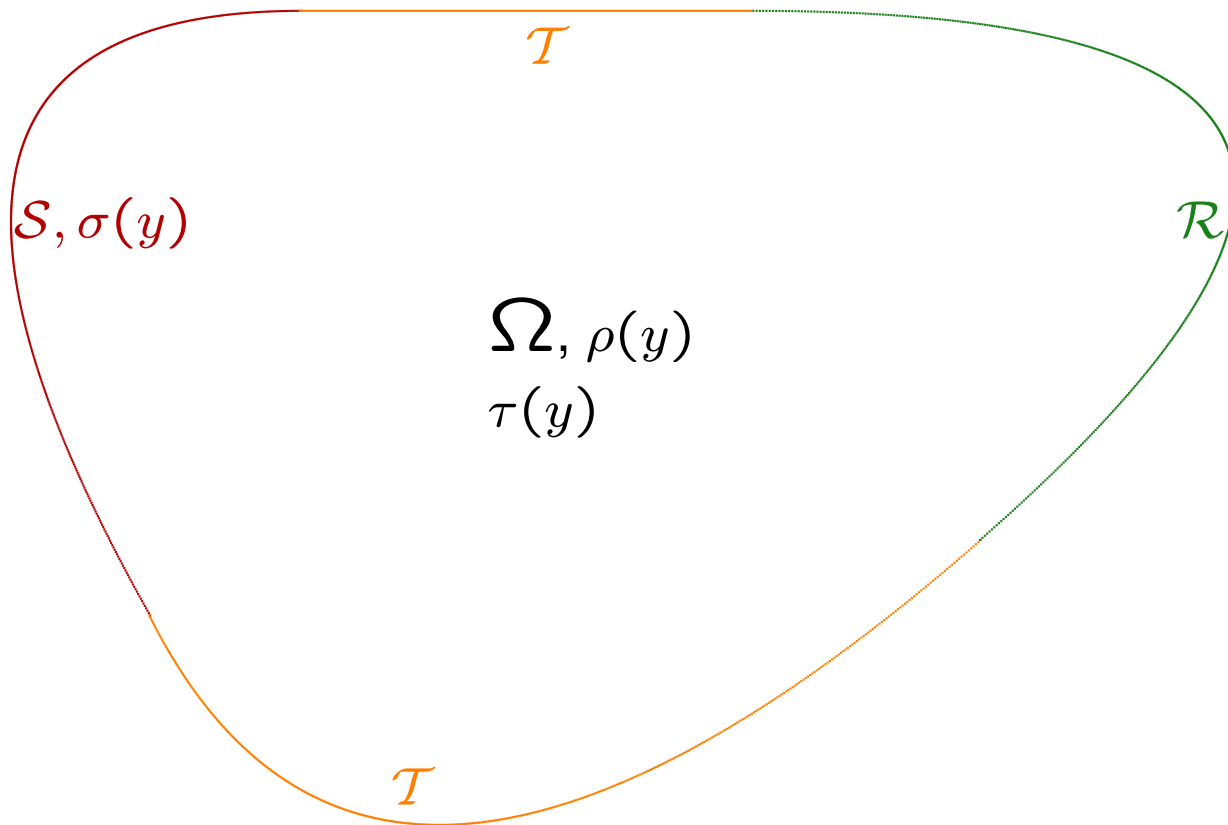
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A routing strategy $p(y)$ is a Wardrop equilibrium if a lone message traveling to \mathcal{R} minimizes the travel time by following $x(y) = p(y)/\|p(y)\|$.

Data



Wardrop equilibrium

Let a message originate in $y_0 \in \mathcal{Q} \cup \Omega$, and reach \mathcal{R} in y_1 . Let s be the curvilinear abscissa along the path.

$$\frac{dy}{ds} = x(s), \quad G = \int_{y_0}^{y_1} \tau(y(s)) \|q(y(s))\| ds.$$

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The H.-J.-B. equation of the optimization problem is

$$\begin{aligned} \forall y \in \Omega, \quad \min_{\|x\|=1} \langle \nabla V(y), x \rangle + \tau(y) \|p(y)\| &= 0, \\ \forall y \in \mathcal{R}, \quad V(y) &= 0. \end{aligned}$$

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The optimum is obtained at $x = -\nabla V / \|\nabla V\|$ and $\|\nabla V\| = \tau \|p\|$.

Computing the Wardrop equilibrium

A vector field q is admissible if $\forall y \in \mathcal{Q}, \langle q(y), n(y) \rangle = -\sigma(y)$,
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Recapitulating the Wardrop conditions yields

$$\begin{aligned}\forall y \in \mathcal{Q}, \quad \langle \nabla V(y), p(y) \rangle &= -\sigma(y), \\ \forall y \in \mathcal{R}, \quad V(y) &= 0 \\ \forall y \in \Omega, \quad \operatorname{div} \left(\frac{1}{\tau(y)} \nabla V(y) \right) &= \rho(y).\end{aligned}$$

A classical mixed Dirichlet-Neuman elliptic P.D.E.

The classics

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Future

William Sandholm: *Population Games and Evolutionary Dynamics*, MIT Press, to appear p.s.b.n.

Thank you