Branching type processes

and DTNs

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Background on Branching

•19th centuty: concern among Victorians about possible extinction of aristocratic surnames.

•Galton posed this question in the *Educational Times* of 1873. The Reverand Watson replied with a solution. Joint publication of the solution in 1874.

•The G-W process: $X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)}$.

•The G-W process with immigratioon: $X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n$.

•Multitype: Y_n is a vector, $Y_{n+1} = A_n(Y_n) + B_n$, where

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n)$$

Example 1: discrete branching with migration

Queue with Vacations, Gated Regime

• $M/G/1/\infty$ queue,

• Arrival rate λ, i.i.d. service times $\{D_n\}$ with first and second moments d, $d^{(2)}$.

• Sequence of vacations: V_n. Will be assumed stationary ergodic, with first and second moments v, $v^{(2)}$.

• Gated regime: at the nth end of vacation, a gate is closed (nth polling instant). Then the server goes on serving the customers present at the queue at that polling instant:
Then the server leaves on vacation.

•We denote:

- B_n:= the number of arrivals during the nth vacation.
- $\xi_h^{(i)}$:= the number of arrivals during the service time of a customer

•Then:

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n, \qquad n \geq n_0.$$

•**Divisibility property:** Denote

$$A_n(x) = \sum_{i=1}^{x} \xi_n^{(i)}$$

Then A_n are nonnegative and divisible:

$$A_n(x+y) = A_n^{(1)}(x) + A_n^{(2)}(y)$$

where $A_n^{(i)}$ are i.i.d.

We use this property do define branching process. Applies for non-integers!

Example 2: continuous branching with migration

Queue with Vacations, Gated Regime

•Define the time to serve N customers as:

$$\tau(N) := \sum_{i=1}^{N} D_i$$

•Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T.

•Denote by $\hat{\mathcal{A}}_n(C_n) = \tau(\mathcal{N}(C_n))$, i.e. the sum of service times of all the arrivals during C_n.

•We obtain

$$C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}. \quad (1)$$

Application: Message Ferry [Kavitha + EA]

- A bus has a given fixed trajectory on the plain.

- It starts at a BS (base station), collects packets. It has d stops. At each stop it transmits packets to the closest sensors.

- V_n is total walking time in cycle n. Previous equation holds for cycle time.

- Known as a globally gated polling system.

Queueing with gated vacations $d = 1$

Expected waiting time: (FIFO queue)

Consider an arbitrary customer. Upon arrival, it has to wait for

1. The residual cycle time C_{res},
2. The service time of all the customers that arrived during C_{past} which is the past cycle time: $d(\lambda E[C_{past}]) = \rho E[C_{past}]$

We have from [Baccelli & Brémaud, 1994]

$$E[C_{res}] = E[C_{past}] = \frac{E[C_0^2]}{2E[C_0]}.$$

Thus the expected waiting time of an arbitrary customer is given by

$$E[W_n] = (1 + \rho) \frac{E[C_0^2]}{2E[C_0]},$$

The expected number of customers in queue in stationary regime (not including service) is obtained using Little's Theorem: $\lambda E[W_n]$.

Conclusion: we need to compute $E[C_0]$ and $E[C_0^2]$!

1 Steady State Probabilities of CBP

Iterating $Y_{n+1} = A_n(Y_n) + B_n$, we obtain from A1:

$$\begin{aligned} Y_2 &= A_1(Y_1) + B_1 \\ &= A_1(A_0(Y_0) + B_0) + B_1 \\ &= A_1^{(0)}(A_0(Y_0)) + A_1^{(1)}(B_0) + B_1 \\ &= A_1^{(0)} A_0^{(0)}(Y_0) + A_1^{(1)}(B_0) + B_1. \end{aligned}$$

$$\begin{aligned} Y_3 &= A_2(Y_2) + B_2 \\ &= A_2(A_1(Y_1) + B_1) + B_2 \\ &= A_2(A_1(A_0(Y_0) + B_0) + B_1) + B_2 \\ &= A_2^{(0)} A_1^{(0)} A_0^{(0)}(Y_0) + A_2^{(1)} A_1^{(1)}(B_0) + A_2^{(2)}(B_1) + B_2 \end{aligned}$$

$$Y_n = \sum_{j=0}^{n-1} \left(\prod_{i=n-j}^{n-1} A_i^{(n-j)} \right) (B_{n-j-1}) + \left(\prod_{i=0}^{n-1} A_i^{(0)} \right) (Y_0), \quad n > 0 \tag{2}$$

(we understand $\prod_{i=n}^{k} A_i(x) = x$ whenever $k < n$, and $\prod_{i=n}^{k} A_i(x) = A_k A_{k-1} \ldots A_n$ whenever $k > n$).

•Under fairly general assumptions, $\lim_{n\to\infty}\left(\prod_{i=0}^{n-1} A_i^{(0)}\right)(y) = 0$, so Y_n has a limit as $n \to \infty$ distributed like

$$Y_n^* =_d \sum_{j=0}^{\infty}\left(\prod_{i=n-j}^{n-1} A_i^{(n-j)}\right)(B_{n-j-1}), \qquad n \in Z, \tag{3}$$

where for each integer i, $\{A_i^{(j)}(\cdot)\}_j$ have the same distribution as $A_i(\cdot)$.

•Sufficient condition: stationarity plus $\|\mathcal{A}\| < 1$.

•Branching processes: $\{A_i^{(j)}(\cdot)\}_j$ are i.i.d.

•Stochastic differential equations: they are equal.

•The representation holds for general dependence: Semi linear processes.

Moments:

•(i) The first moment of X_n^* is given by

$$E[X_0^*] = (I - \mathcal{A})^{-1} b. \tag{4}$$

•(ii) Assume that the first and second moments b_i and $b_i^{(2)}$'s are finite and that F satisfies

$$\lim_{n \to \infty} F^n = 0. \tag{5}$$

Define Q to be the matrix whose ijth entry is $Q_{ij} = \sum_{k=1}^{d} \overline{y}_k \Gamma^{(k)}$. Then the matrix $cov(X^*)$ is the unique solution of the set of linear equations:

$$cov(X) \quad = \quad cov(B) + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \widehat{\mathcal{B}}(r) + \left[\mathcal{A}^r \widehat{\mathcal{B}}(r) \right]^T \right) + F(cov[X]) + Q. \tag{6}$$

The second moment matrix $E[XX^T]$ in steady state is the unique solution of the set of linear equations:

$$E[XX^T] \quad = \quad E[B_0 B_0^T] + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \mathcal{B}(r) + \left[\mathcal{A}^r \mathcal{B}(r) \right]^T \right) + F(E[XX^T]) + Q. \tag{7}$$

2 Example: Delay Tolerant Ad-hoc Networks

- Delay tolerant Ad-hoc Networks make use of nodes' mobility to compensate for lack of instantaneous connectivity.

- Information sent by a source to a disconnected destination can be forwarded and relayed by other mobile nodes.

- Let X_n^+ be the number of nodes that have a copy of the packet at time n,

- Let X_n^- be the number of nodes that do not have a copy of the packet at time n.

- Mobility: a mobile present at time n may leave and other may arrive. Let B_n be the number of new arrivals.

- Let $\rho_n^{(i)}$ and $\hat{\rho}_n^{(i)}$ be the indicator that node i remains in the system for the next slot. ρ is used for nodes that have the packet and $\hat{\rho}$ for the others.

- Let $\xi_n^{(i)}$ be the indicator that the source meats mobile i at time slot n. These are i.i.d. Then

$$X_{n+1}^+ = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^- = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

Controling the Energy

- Assume that the source limits the transmissions in order to save energy
- Let ζ_n be the indicator that the source intends to transmit a packet at time n. Assume ζ_n are i.i.d.

$$X_{n+1}^{+} = \sum_{i=1}^{X_n^{+}} \rho_n^{(i)} + \zeta_n \sum_{i=1}^{X_n^{-}} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^{-} = \sum_{i=1}^{X_n^{-}} \hat{\rho}_n^{(i)} (1 - \zeta_n \xi_n^{(i)}) + B_n$$

- This is a semi-linear process, not a branching process

Partial Information

•Observations: Assume that each node is "sampled" each time unit with some small probability.

•Let $\alpha_n^{(i)}$ be the indicator that node i is sampled at time n.

$$X_{n+1}^+ = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^- = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

$$Y_{n+1} = \sum_{i=1}^{X_n^+} \alpha_n^{(i)}$$

Filtering

- Objective: monitoring the number of packets.
- Example: first order linear filter
- Let $\widehat{X}_n^+$ be the estimator of X_n^+, The estimation error is $\epsilon_n = \hat{X}_n^+ - X_n^+$.

$$X_{n+1}^+ = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^- = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

$$Y_{n+1} = \sum_{i=1}^{X_n^+} \alpha_n^{(i)}$$

$$\hat{X}_{n+1} = K_1(Y_n - \hat{X}_n) + K_2 \hat{X}_n$$

$$\epsilon_n = \hat{X}_n - X_n$$

- Semi-linear process. We can compute $E[\epsilon_n^2]$ and compute K that minimizes it.